Midterm - Probability II (2024-25)
Time: 3 hours.
Attempt all questions. The total marks is 30.

1. The random variables X and Y have joint density function

$$f(x,y) = 12xy(1-x) \qquad 0 < x < 1,\, 0 < y < 1,$$

and equal to 0 otherwise.

   (a) Are X and Y independent? **[1 mark]**

   (b) Find $E(X)$. **[2 marks]**

   (c) Find $P(X+Y \leq 1)$. **[2 marks]**

2. Consider a simple random walk S_n on $\mathbf{Z}$ starting at 0 with equal probability of going to the right or to the left. That is, $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$, where the X_i's are independent random variables taking the values $+1$ or -1 with probability $\frac{1}{2}$ each. Let $a > 0$ and $b > 0$ be integers. What is the probability that $S_1 > -b, \cdots, S_{n-1} > -b$, $S_n = a$? **[5 marks]**

3. Consider a simple random walk on $\mathbf{Z}$ with probability of going to the right equal to 0.6 and probability of going to the left equal to 0.4. Suppose the walk starts at 50. What is the probability that the walk hits 0 before 100? **[5 marks]**

4. In a sequence of Bernoulli trials with success (S) probability p and failure (F) probability $q = 1-p$, let a_n be the probability that the combination SF occurs for the first time at trials $n-1$ and n. Let T be the random variable such that $P(T = n) = a_n$.

   (a) Find the generating function of T. **[5 marks]** HINT: condition on first trial.

   (b) Find the mean of T. **[2 marks]**

5. (a) For a nonnegative integer valued random variable Y show that

$$\mathbb{E}Y = \sum_{k=0}^{\infty} P(Y > k). \quad \textbf{[3 marks]}$$

   (b) Let $\{X_n\}$ be a sequence of mutually independent integer-valued random variables with a common distribution. Suppose that the X_n do not have a finite expectation, that is $\mathbb{E}|X_n| = \infty$, and let $A > 0$ be a positive constant. Show that the probability is one that infinitely many among the events $\{|X_n| > An\}$ occur. **[5 marks]**
   HINT: Use the second Borel-Cantelli lemma: If B_n is a sequence of independent events such that $\sum P(B_n) = \infty$ then $P($ infinitely many B_n occur$) = 1$.